

# Describing relativistic fermions with two-component field

Yu-Qi Chen

*Institute of Theoretical Physics, Academia Sinica, Beijing 100080*

## Abstract

With a non-unitary transformation, the Lagrangian of a Dirac fermion is decomposed into two decoupled sectors. We propose to describe massive relativistic fermions in gauge theories in a two-component form. All relations between the Green's functions in this form and those in the conventional four-component form are derived. It is shown that the  $S$ -matrix elements in both forms are exactly the same. The description of the fermion in the new form simplifies significantly the  $\gamma$ -matrix algebra in the four-component form. In particular, in perturbative calculations the propagator of the fermion is a scalar function. The advantages of the two-component form make it very useful in practical applications.

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Describing relativistic fermions using four-component spinors is one of the most fantastic inventions made by Dirac[1]. In this description, a fermion and its anti-particle are put in the same four dimensional spinor representation of the Lorentz group. The motion of the fermion satisfies the Dirac equation, and the Lagrangian expressed in this field keeps manifest Lorentz invariance and the local interaction terms. The whole formalism is simple and elegant. However, it is not very convenient in practical applications. For instance, in evaluating Feynman diagrams of multi-particle processes, handling the  $\gamma$ -matrix algebra turns out to be very hard and tedious, even by virtue of modern computers. In solving bound state problems, the complicated spinor structure of the wave function makes solutions difficult to gain. In lattice simulations, the discretized Lagrangian of chiral fermions suffers from the doubling problem. In order to overcome all these drawbacks, it is crucial to develop new methods to describe relativistic fermions.

In this paper, we propose a two-component form to describe massive relativistic fermions. We establish the formalism in the following way. First with a non-unitary transformation, we show that the Lagrangian of the fermion is decomposed into two decoupled sectors, and each of them depends on a two-component field. One of them is in a nonlocal form similar to the unexpanded Lagrangian in the Heavy Quark Effective Theory (HQET), while the other one is trivial. We then derive relationships between the Green's functions in this two-component form and those in the conventional four-component form. Finally, with one of these relationships, we show that the  $S$ -matrix elements in both forms are exactly the same. The conclusion holds not only for fermion scattering processes but also for fermion antifermion annihilation processes. In this way, we show that a relativistic massive fermion and antifermion can be uniformly described with a two-component field.

In history, a two-component field was proposed to describe a nonrelativistic fermion by Pauli. In nonrelativistic limit the fermion and the antifermion near on-mass shell are decoupled so that they can be described by separate two-component fields. This strategy has been taken in constructing HQET[2] and nonrelativistic QCD ( NRQCD )[3]. In extreme-relativistic case, the left hand and the right hand fermions are decoupled and each of them can be described by a two-component field. At first sight, it seems surprising to describe a massive relativistic fermion and anti-fermion simultaneously with a single two-component field. However, it is not difficult to see from the following facts: The generic  $S$ -matrix elements for a fermion scattering or fermion-antifermion annihilation processes

can be written as a bilinear form of the Dirac spinors with  $4 \times 4$  matrixes in between. The free four-component wavefunction satisfying the Dirac equation can be expressed by the Pauli two-component wavefunction. Thus eventually the  $S$ -matrix elements can be reduced to a bilinear form of the Pauli spinors with  $2 \times 2$  matrixes in between. In the two-component form, we calculate the  $S$ -matrix elements directly from the two-component Lagrangian and the wavefunctions from the beginning. Here the fermion and the antifermion are distinguished via different wavefunctions.

The description of the fermions in this form has a number of advantages. An apparent one is that it significantly simplifies the  $\gamma$ -matrix algebra. Instead of the  $4 \times 4$   $\gamma$  matrixes, only the  $2 \times 2$   $\sigma$  matrixes appear in the Lagrangian of the two-component form. Moreover, in perturbative calculations, the propagator of the free fermion is a scalar and the interaction vertices can be expressed as a  $2 \times 2$  matrix. It is very convenient to carry out numerical calculations for complicated high energy processes.

We start the derivation from the following observation. For a generic Hermitian matrix  $H$ ,

$$H = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}, \quad (1)$$

there exists a non-unitary transformation expressed by a triangle matrix  $S$ ,

$$S = \begin{pmatrix} I & 0 \\ -C^{-1}B^\dagger & I \end{pmatrix}, \quad (2)$$

which makes  $S^\dagger H S$  diagonalized. Specifically, we have:

$$S^\dagger H S = \begin{pmatrix} A - BC^{-1}B^\dagger & 0 \\ 0 & C \end{pmatrix}. \quad (3)$$

In a gauge theory, the Lagrangian density of a fermion is given by

$$\mathcal{L} = \bar{\Psi}(x) (i \not{D} - m) \Psi(x), \quad (4)$$

where  $D_\mu = \partial_\mu - igA_\mu$  is the covariant derivative. To make the transformation on the Lagrangian density, we follow the technique adopted in HQET and decompose the Dirac four-component field  $\Psi(x)$  as[2]:

$$\Psi(x) = \Psi_+(x) + \Psi_-(x), \quad (5)$$

where  $\Psi_{\pm}(x) = P_{\pm} \Psi(x)$  and the projection operators  $P_{\pm}$  are defined by

$$P_{\pm} \equiv \frac{1 \pm \not{V}}{2}, \quad (6)$$

with the four-velocity parameter  $V$  satisfying  $V^2 = 1$ . The projection operators divide the four dimensional spinor space into two two-dimensional spinor subspaces. With this decomposition, the matrix  $L \equiv i \not{D} - m$  can be rewritten as:

$$L = \begin{pmatrix} iD \cdot V - m & iD \cdot \sigma_{\perp} \\ iD \cdot \bar{\sigma}_{\perp} & -iD \cdot V - m \end{pmatrix}, \quad (7)$$

where  $\sigma^{\mu} \equiv (1, \boldsymbol{\sigma})$  and  $\bar{\sigma}^{\mu} \equiv (1, -\boldsymbol{\sigma})$ ;  $\sigma_{\perp}^{\mu} \equiv \sigma^{\mu} - \sigma \cdot V V^{\mu}$  and  $\bar{\sigma}_{\perp}^{\mu} \equiv \bar{\sigma}^{\mu} - \bar{\sigma} \cdot V V^{\mu}$ .

According to Eq. (2), choosing the transformation matrix  $S$  as:

$$S = \begin{pmatrix} \text{I} & 0 \\ \frac{1}{iD \cdot V + m} iD \cdot \bar{\sigma}_{\perp} & \text{I} \end{pmatrix}, \quad (8)$$

and defining  $\bar{S} \equiv \gamma^0 S^{\dagger} \gamma^0$ ,  $L$  transforms into:

$$\begin{aligned} L' &= \bar{S} L S \\ &= \begin{pmatrix} iD \cdot V - m - D \cdot \sigma_{\perp} \frac{1}{iD \cdot V + m} D \cdot \bar{\sigma}_{\perp} & 0 \\ 0 & -iD \cdot V - m \end{pmatrix}. \end{aligned} \quad (9)$$

Correspondingly, the field  $\Psi(x)$  transforms into  $\Psi'(x) = S^{-1} \Psi(x)$ . Its component form reads:

$$\begin{pmatrix} \Psi'_+(x) \\ \Psi'_-(x) \end{pmatrix} = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) - \Psi_-^c(x) \end{pmatrix}, \quad (10)$$

where

$$\Psi_-^c(x) = \frac{1}{iD \cdot V + m} iD \cdot \bar{\sigma}_{\perp} \Psi_+(x). \quad (11)$$

We see that  $\Psi'_-(x) = 0$  if  $\Psi_-(x)$  satisfies the Dirac equation.

The Lagrangian density of the fermion can then be written in a compact form as:

$$\mathcal{L} = \bar{\Psi}'(x) L' \Psi'(x). \quad (12)$$

Eqs. (9)-(12) imply that after the transformation, the fermion Lagrangian density is decomposed into two independent sectors. They are decoupled, and each of them is described

by a two-component field. It is interesting that the first sector is similar with the nonlocal form of HQET which was derived using the equation of motion or the generating functional method. In these approaches, certain approximations have been made. Here there is no approximation being made when we derive it via the non-unitary transformation (8). This ensures the equivalence of the four-component form and the two-component form.

The first term is invariant under the reparameterization transformation[4]. This ensures that the physical predictions made from it are independent of the choice of the velocity parameter  $V$  in spite of an explicit  $V$ -dependence in the decomposition (5).

The original Lagrangian density (4) can be recovered by the inverse transformation:

$$L = \bar{S}^{-1} L' S^{-1}, \quad \Psi(x) = S \Psi'(x). \quad (13)$$

The transformation leads to the following field identity:

$$\Psi(x) \bar{\Psi}(y) = S(x) \Psi'(x) \bar{\Psi}'(y) \bar{S}(y). \quad (14)$$

This identity implies a set of relations between the Green's functions in both forms. With  $S(x)$  given in Eq.(8), it follows that:

$$P_+ \langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle^B P_+ = \langle 0 | T \Psi'_+(x) \bar{\Psi}'_+(y) | 0 \rangle^B, \quad (15)$$

$$P_- \langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle^B P_+ = \langle 0 | T \Psi_-^c(x) \bar{\Psi}'_+(y) | 0 \rangle^B, \quad (16)$$

$$\begin{aligned} P_- \langle 0 | T \Psi(x) \bar{\Psi}(y) | 0 \rangle^B P_- &= \langle 0 | T \Psi'_-(x) \bar{\Psi}'_-(y) | 0 \rangle^B \\ &- \langle 0 | T \Psi_-^c(x) \bar{\Psi}_-^c(y) | 0 \rangle^B, \end{aligned} \quad (17)$$

where we have omitted the conjugation relation of (16). The superscript  $B$  denotes that they are the Green's functions in an arbitrary background field. With the path integral method, any Green's function with gluon external lines can be obtained by functional derivative with respect to the background field  $B(x)$ . Since the determinant of the transform (8) is unity, the measure of the integration element of the fermion field remains unchanged. This ensures the validation of these relations beyond the tree level. From these relations we see that once the right hand side of each relation is calculated in the two-component form, all possible projected Green's functions in the original theory can be obtained, and *vice versa*. This implies that both forms are equivalent. Thus these relations signify connections between the four-component form and the two-component one.

In HQET, relation (15) has been used as matching conditions[5] to determine the renormalized effective Lagrangian. Relation (16) has been used as matching conditions[6] to determine the renormalized transformation of the heavy quark field in the reparametrization invariance. The relation (17) is a new one, which has not yet been derived before.

We now show that the  $S$ -matrix elements evaluated with both forms are the same. For a process of a fermion scattered by arbitrary gluons or fermion anti-fermion annihilation or creation, a generic form of the Green's functions in the momentum space can be expressed as:

$$G(p, p'; \Gamma) = G(p) \Gamma G(p'), \quad (18)$$

where  $p$  and  $p'$  are the external momenta of the fermions;  $G(p)$  and  $G(p')$  are the propagators of the free fermions;  $\Gamma$  denotes the interaction vertex. Using the reduction formula, the  $S$ -matrix element for the fermion scattering process in the four-component form can be obtained as follows:

$$\begin{aligned} T_{ss'}(p' \rightarrow p + X) &= \lim_{p^2, p'^2 \rightarrow m^2} \bar{u}_s(p) G^{-1}(p) G(p, p'; \Gamma) G^{-1}(p') u_{s'}(p') \\ &= \bar{u}_s(p) \Gamma u_{s'}(p'), \end{aligned} \quad (19)$$

where  $u_s(p)$  and  $u_{s'}(p')$  are the four-component Dirac wave functions of the fermions. Similarly, the  $S$ -matrix element for the fermion anti-fermion annihilation process reads

$$\begin{aligned} T_{ss'}(\bar{p} + p' \rightarrow X) &= \lim_{p^2, p'^2 \rightarrow m^2} \bar{v}_s(p) G^{-1}(-p) G(-p, p'; \Gamma) G^{-1}(p') u_{s'}(p') \\ &= \bar{v}_s(p) \Gamma u_{s'}(p'), \end{aligned} \quad (20)$$

where  $v_s(p)$  is the Dirac wavefunction of the antifermion.

To relate the  $S$ -matrix in the four-component form to that in the two-component form, we first write down the wavefunction of the fermion in the two-component description. For simplicity, we take the four-velocity parameter  $V = (1, 0, 0, 0)$ . However, all results can easily be generalized to an arbitrary  $V$ .

$$\tilde{u}_s(p) = P_+ u_s(p) = \sqrt{E + m} \xi_s \quad (21)$$

$$\tilde{v}_s(p) = P_+ v_s(p) = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{E + m}} \xi_s, \quad (22)$$

where  $E \equiv p_0$ , and  $\xi_s$  ( $s = 1, 2$ ) are the Pauli two-component spinors.

The two-component wavefunctions are not orthogonal and normalized. This is not surprising since a non-unitary transformation has been performed on the fermion field. Eqs. (21)

(22) imply that the wavefunctions of the fermion and the antifermion are different. It can be used to distinguish the fermion and the antifermion.

We notice that for antifermions,  $\mathbf{p} = 0$  is a singular point. However, the singularity cancels in the final expression.

The free propagator of the fermion in the two-component form  $\tilde{G}(p)$  can be read from the Lagrangian:

$$\tilde{G}(p) = i \frac{E + m}{p^2 - m^2 + i\epsilon} . \quad (23)$$

Two poles correspond to the contributions from the fermion and the antifermion. In fact, this propagator can be decomposed as:

$$\tilde{G}(p) = \frac{i}{2\omega} \left( \frac{\sum_s \tilde{u}_s(\tilde{p}) \bar{\tilde{u}}_s(\tilde{p})}{E - \omega + i\epsilon} + \frac{\sum_s \tilde{v}_s(\tilde{p}) \bar{\tilde{v}}_s(\tilde{p})}{E + \omega - i\epsilon} \right) , \quad (24)$$

where  $\omega \equiv \sqrt{\mathbf{p}^2 + m^2}$ ,  $\tilde{p} \equiv (\omega, \mathbf{p})$ . This is just the result that we expect to have.

With the  $\tilde{G}(p)$  given in (23), we can show the following identities:

$$\lim_{p^2 \rightarrow m^2} G(p) P_+ \tilde{G}^{-1}(p) \tilde{u}_s(p) = u_s(p) , \quad (25)$$

$$\lim_{p^2 \rightarrow m^2} G(-p) P_+ \tilde{G}^{-1}(-p) \tilde{v}_s(p) = v_s(p) . \quad (26)$$

Similar to Eqs. (19) and (20), the  $S$ -matrix element for the fermion scattering process in the two-component form reads

$$\tilde{T}_{ss'}(p' \rightarrow p + X) = \lim_{p^2, p'^2 \rightarrow m^2} \bar{\tilde{u}}_s(p) \tilde{G}^{-1}(p) \tilde{G}(p, p'; \Gamma) \tilde{G}^{-1}(p') \tilde{u}_{s'}(p') , \quad (27)$$

and that for the annihilation process reads

$$\tilde{T}_{ss'}(p' + p \rightarrow X) = \lim_{p^2, p'^2 \rightarrow m^2} \bar{\tilde{v}}_s(p) \tilde{G}^{-1}(-p) \tilde{G}(-p, p'; \Gamma) \tilde{G}^{-1}(p') \tilde{u}_{s'}(p') , \quad (28)$$

where  $\tilde{G}(p, p'; \Gamma)$  is the Green's function evaluated in the two-component form. The relation between the Green's functions (15) implies that

$$\tilde{G}(\pm p, p'; \Gamma) = P_+ G(\pm p, p'; \Gamma) P_+ . \quad (29)$$

Imposing this equation and Eqs. (25) and (26) on Eqs. (27) and (28), we find that

$$\tilde{T}_{ss'}(p' \rightarrow p + X) = T_{ss'}(p' \rightarrow p + X) , \quad (30)$$

$$\tilde{T}_{ss'}(p' + p \rightarrow X) = T_{ss'}(p' + p \rightarrow X) . \quad (31)$$

Thus, from the relations (15), we have shown that the  $S$ -matrix elements both for scattering processes and for annihilation processes are the same in both formulations. This demonstration can easily be generalized to the cases of multi-fermionic external lines.

Obviously, the two-component form simplifies significantly the  $\gamma$ -matrix algebra in the conventional four-component form. Instead of the  $4 \times 4$   $\gamma$  matrixes, only the  $2 \times 2$   $\sigma$  matrixes are involved in the former case. This feature makes it very useful in practical applications. The calculations for the  $T$ -matrix elements of the processes with multi-external lines are very complicated. The helicity amplitude method[7] and the direct amplitude method[8] have been proposed to simplify the calculations. However, these methods are efficient only for massless fermions. The two-component form presented here is equally well both for massless fermions and for massive fermions. The perturbative calculations can easily be carried out in the axial gauge  $A \cdot V = 0$ . In this gauge, the second sector is a free term without any interaction. Hence all the contributions to the  $S$ -matrix elements arise from the first sector. The Lagrangian of this term reads:

$$\tilde{L} = \bar{\Psi}_+(x) \left( i\partial_0 - m + \mathbf{D} \cdot \boldsymbol{\sigma} \frac{1}{i\partial_0 + m} \mathbf{D} \cdot \boldsymbol{\sigma} \right) \Psi_+(x). \quad (32)$$

The Feynman rules can easily be obtained. The propagator of the free fermion has been given in (23). Here we list the Feynman rules of the interaction vertices as follows:

$$i g_s t^a \left( \frac{\boldsymbol{\sigma} \cdot \mathbf{p}_1 \sigma^i}{E_1 + m} + \frac{\sigma^i \boldsymbol{\sigma} \cdot \mathbf{p}_2}{E_2 + m} \right)$$

$$-i g_s^2 \left( t^a t^b \frac{\sigma^i \sigma^j}{E_1 + k_1^0 + m} + t^b t^a \frac{\sigma^j \sigma^i}{E_1 + k_2^0 + m} \right)$$

With these rules and (21), (22), and (23), perturbation calculations can be carried out in the two-component form. Since the propagator of fermion given in eq. (23) is a scalar and the most generic interaction vertex can be written as  $a + b_i \sigma_i$ , whose product can be easily dealt with, we expect that the numeric perturbation calculations can be significantly simplified.

Another possible application is to solve the bound state problem. In the two-component form, the simpler spinor structure of the wavefunction and the equations makes bound state problems easier to be solved.

The two-component form might also be used to formulate chiral fermion on Lattice to overcome the doubling problem[9].

In conclusion, we show that the relativistic massive fermions in gauge theories can be described with a two-component form. In this form, the fermion and the anti-fermion are uniformly expressed as the two-component field instead of the usual Dirac four-component field. We have shown that the  $S$ -matrix elements in this form are exactly the same as that in the conventional four-component form. The two-component form simplifies significantly the  $\gamma$ -matrix algebra in the four-component form. It will be very useful in calculating the high energy processes and in solving the bound state problems. We also expect that it is useful in simulating the chiral fermions on the lattice.

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